Maxwell's Equations → Griffiths Tomes

(i) $\nabla \cdot E = \frac{1}{\epsilon_0}\rho$ --- (i)

(ii) $\nabla \cdot B = 0$ --- (ii)

(iii) $\nabla \times E = -\frac{\partial B}{\partial t}$ --- (iii)

(iv) $\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$ --- (iv) → Ampere's law with Maxwell's correction

Together with the force law
$F = q(E + v \times B)$, they summarise
the entire theoretical content of classical electrodynamics

Even the continuity eqn

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

↳ mathematical expression of conservation of charge

Can be derived from the Maxwell's eqn by applying divergence to eqn (iv)

E → can be produced either by charges (ρ)
or by changing magnetic fields $\left(\frac{\partial B}{\partial t}\right)$

B → can be produced either by currents (J)
or by changing electric fields $\left(\frac{\partial E}{\partial t}\right)$

It is preferable to write

$$\nabla \cdot E = \frac{1}{\epsilon_0}\rho \qquad \nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla \cdot B = 0 \qquad \nabla \times B - \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \mu_0 J$$

(fields) E & B on left and sources (ρ and J) on the right

E-M fields → attributable to charges and currents

Maxwell's eqn → how charges produce fields

Force law → how fields affect charges

Maxwell's Equations in Matter

Materials → subject to electric and magnetic polarization.

Inside polarized matter → accumulations of "bound" charge and current → no direct control

From static case → an electric polarization P produces a bound charge density

$$\rho_b = -\nabla \cdot P \quad --- (1)$$

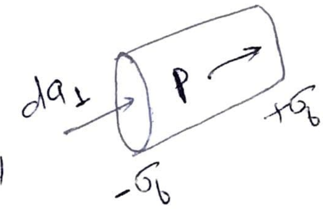
A magnetic polarization (or magnetization) M results in a bound current

$$J_b = \nabla \times M \quad --- (2)$$

One new feature to consider is the nonstatic case:

Any change in the electric polarization involves a flow of (bound) charge (call it J_p), which must be included in the total current

We examine a tiny chunk of polarized material



Polarization introduces a charge density $\sigma_p = P$ at one end and $-\sigma_b$ at the other. If P now increases a bit, the charge on each end increases accordingly, giving a net current

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp$$

The current density, therefore is

$$J_p = \frac{\partial P}{\partial t} \quad --- (3)$$

This Polarization current has nothing to do with the bound current J_b.

Bound current → associated with magnetization of the material and involves the spin and orbital motion of electrons

J_p → result of the linear motion of charge when the electric polarization changes.

We check that eqn (3) is consistent with the continuity eqn?

$$\nabla \cdot J_p = \nabla \cdot \frac{\partial P}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot P) = -\frac{\partial \rho_b}{\partial t}$$

Yes ✓

The total charge density can be separated into two parts

$$\rho = \rho_f + \rho_b$$
$$= \rho_f - \nabla \cdot P \quad \text{--- (4)}$$

and the current density into three parts

$$J = J_f + J_b + J_p$$
$$= J_f + \nabla \times M + \frac{\partial P}{\partial t} \quad \text{--- (5)}$$

Gauss's Law now

$$\nabla \cdot E = \frac{1}{\varepsilon_0}(\rho_f - \nabla \cdot P)$$

or $$\nabla \cdot D = \rho_f \quad \text{--- (6)}$$

where D, as in static case, is given by

$$\boxed{D = \varepsilon_0 E + P} \quad \text{--- (7)}$$

Meanwhile, Ampere's law (with Maxwell's term)

$$\nabla \times B = \mu_0\left(J_f + \nabla \times M + \frac{\partial P}{\partial t}\right) + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

or $\nabla \times H = J_f + \frac{\partial D}{\partial t}$ --- (8)

where

$$H \equiv \frac{1}{\mu_0} B - M \quad \text{---} (9)$$

Faraday's law & $\nabla \cdot B = 0$ → not affected by our separation of charge and current into free & bound parts, since they do not involve ρ or J.

In terms of free charges and currents, Maxwell eqⁿs

(i) $\nabla \cdot D = \rho_f$ (iii) $\nabla \times E = -\frac{\partial B}{\partial t}$ --- (10)

(ii) $\nabla \cdot B = 0$ (iv) $\nabla \times H = J_f + \frac{\partial D}{\partial t}$

Relations between E and D, B & H depend on the nature of material; for linear media

$$P = \varepsilon_0 \chi_e E \quad \text{and} \quad M = \chi_m H \quad \text{---} (11)$$

$$D = \varepsilon E \quad \text{and} \quad H = \frac{1}{\mu} B \quad \text{---} (12)$$

where $\varepsilon \equiv \varepsilon_0 (1 + \chi_e)$ and $\mu \equiv \mu_0 (1 + \chi_m)$

D → electric displacement

↳ second term in (iv) → displacement current

$$J_d = \frac{\partial D}{\partial t} \quad \text{---} (13)$$

Boundary Conditions

In general, the fields E, B, D and H

↓

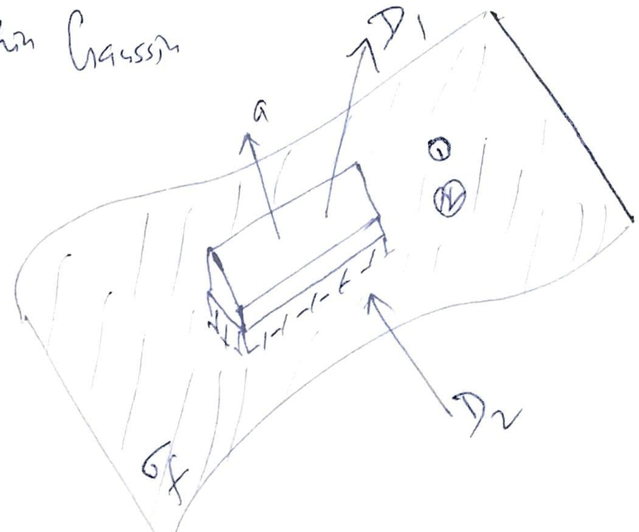
discontinuous at a boundary between two different media or at a surface that carries charge density σ or current density K.

Explicit form of these discontinuities may be deduced from Maxwell's eq^n in their integral form

$$\left.\begin{aligned}&\text{(i)}\ \oint_S D\cdot da = Q_{f\,enc}\\ &\text{(ii)}\ \oint_S B\cdot da = 0\end{aligned}\right\}\ \text{over any closed surface } S.$$

$$\left.\begin{aligned}&\text{(iii)}\ \oint_P E\cdot dl = -\frac{\partial}{\partial t}\int_S B\cdot da\\ &\text{(iv)}\ \oint_P H\cdot dl = I_{f\,enc} + \frac{d}{dt}\int_S D\cdot da\end{aligned}\right\}\ \text{for any surface } S \text{ bounded by the closed loop } P.$$

Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary



$$D_1\cdot a - D_2\cdot a = \sigma_f\, a$$

The edge of the wafer contributes nothing in the limit as the thickness goes to zero, nor does any volume charge density.
The component of D that is perpendicular to the interface is discontinuous in the amount

$$\boxed{D_1^\perp - D_2^\perp = \sigma_f} \quad \text{---} \quad (1)$$

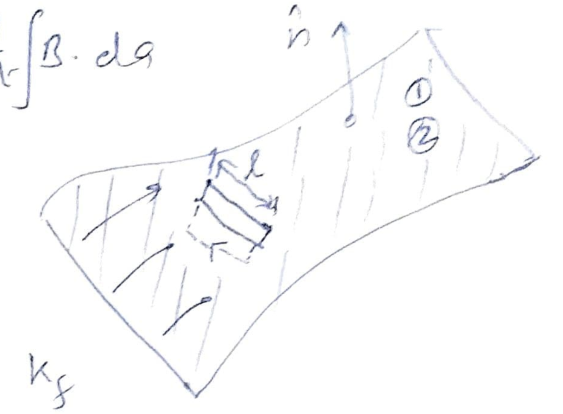
identical reasoning, applied to eq^n (ii)

$$\boxed{B_1^\perp - B_2^\perp = 0} \quad \text{---} \quad (2)$$

Turning to (iii), a very thin Amperian loop straddling the surface gives

$$E_1 \cdot l - E_2 \cdot l = -\frac{d}{dt}\int B \cdot da$$

But in the limit as the width of the loop goes to zero, the flux vanishes



$$\Rightarrow \quad E_1^{\parallel} - E_2^{\parallel} = 0 \quad \text{———(3)}$$

i.e. the component of E parallel to the interface are continuous across the boundary.

By the same token, (iv) implies

$$H_1 \cdot l - H_2 \cdot l = I_{f_{enc}}$$ → free current passing through Amperian loop.

No volume current density will contribute → in the limit of infinitesimal width.

→ but a surface current can.

If $\hat{n}$ → unit vector perpendicular to the interface so $(\hat{n} \times \vec{l})$ → normal to the Amperian loop, then

$$I_{f_{enc}} = K_f \cdot (\hat{n} \times \hat{l}) = (K_f \times \hat{n}) \cdot l,$$

$$\Rightarrow \quad H_1^{\parallel} - H_2^{\parallel} = K_f \times \hat{n} \quad \text{—(4)}$$

↳ free surface current density

eq^n (1)–(4) are the general boundary conditions for electrodynamics. In case of linear media, they can be expressed in terms of E and B alone

(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$ (iii) $E_1^{\parallel} - E_2^{\parallel} = 0$

(ii) $B_1^{\perp} - B_2^{\perp} = 0$ (iv) $\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = K_f \times \hat{n}$ —(5)

If there is no free charge or free current at the interface

(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0$ (iii) $E_1^{\parallel} - E_2^{\parallel} = 0$

(ii) $B_1^{\perp} - B_2^{\perp} = 0$ (iv) $\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = 0$ —(6)

Basis for theory of reflection and refraction.